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A Complex Inversion Theory for Convolution Transforms

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1. INTRODUCTION

The convolution transform with Kernel G is given by

$$f(x) = \int_{-\infty}^{+\infty} G(x - t)\,\phi(t)\,dt. \tag{1.1}$$

If E defined by

$$[E(z)]^{-1} = \int_{-\infty}^{+\infty} e^{-zt} G(t)\,dt \tag{1.2}$$

is an entire function of a special type, then it is a result of Hirschman and Widder, [1], [2], that $E(D)f(x) = \phi(x)$ where $D \equiv d/dx$ and

$$E(D)f(x) = \lim_{\rho \to 1-} \frac{1}{2\pi i} \int_{C_\rho} K(w)\,f(x + \rho w)\,dw. \tag{1.3}$$

$K(w)$ is the Borel transform of $E(z)$ and C_ρ is a certain contour in the complex plane. We refer to this as a complex inversion formula. The restrictions on E imposed in [1], [2] are quite severe and it is the purpose of this paper to relax some of these conditions at the cost of some precision of results.

We will say that an entire function E is in class P if

(i) E is of exponential type,

(ii) E has no zeros along the imaginary axis,

(iii) $\lim_{|y| \to \infty} (1/|y|) \log |E(iy)| = \alpha > 0$.

Thus if E is in class P, then E is an entire function of order 1 and type τ with $\alpha \leqslant \tau < \infty$. If $h(\theta)$ is the Phragmén-Lindelöf indicator function of E defined by

$$h(\theta) = \limsup_{r \to \infty} r^{-1} \log |E(re^{i\theta})|,$$

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then

$$h\left(\pm\frac{\pi}{2}\right) = \alpha. \tag{1.4}$$

Moreover, important to our work is the result of Pólya, [3, p. 74], that

$$E(z) = \frac{1}{2\pi i}\int_C K(w)\, e^{zw}\, dw, \tag{1.5}$$

where $K(w)$ is the above-mentioned Borel transform of $E(z)$ defined by

$$K(w) = \sum_{n=0}^{\infty} E^{(n)}(0)\, w^{-n-1}. \tag{1.6}$$

(Actually we use the analytic continuation of K.) C is any contour containing in its interior the indicator diagram of E (the closed convex hull of the singularities of K). This indicator diagram is a bounded set. In addition, Pólya's theorem asserts the crucial fact that $h(-\theta)$ is the supporting function of this set.

We are led by (1.5) to define

$$E(D) f(x) = \frac{1}{2\pi i}\int_C K(w) f(x+w)\, dw.$$

Following the work of Hirschman and Widder however, we find it more useful to define

$$E(D) f(x) = \lim_{\rho\to 1-} \frac{1}{2\pi i}\int_{C_\rho} K(w) f(x+\rho w)\, dw. \tag{1.7}$$

We postpone the precise definition of C_ρ to a later section.

The work in [1], [2] corresponds to the case in which the indicator diagram of E is a line segment of the form $[-i\alpha, i\alpha]$.

Our main results are contained in Section 3 while in Section 4, we relax slightly the requirement on E expressed by (iii) in the definition of class P.

2. Preliminary Results

In this section, we obtain some needed results on the kernel G and the function f. As indicated earlier, we begin with the entire function E. Define

$$G(w) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{itw}[E(it)]^{-1}\, dt \tag{2.1}$$

with $w = u + iv$.

LEMMA 2.1. *Let E be an entire function in class P. Then*

(a) *for every $\epsilon > 0$ and every positive integer n,*

$$E^{(n)}(iy) = O(e^{|y|(\alpha+\epsilon)}), \qquad y \to \pm\infty,$$

(b) *$G(w)$ is analytic in the strip $|v| < \alpha$,*

(c) *for every positive integer n,*

$$G(u+iv) = O(|u|^{-n}), \qquad u \to \pm\infty,$$

uniformly in every substrip $|v| < \alpha - \epsilon$, $(0 < \epsilon < \alpha)$, of the strip $|v| < \alpha$.

$E^{(n)}(z)$ is an entire function of exponential type and moreover its indicator diagram is a subset of the indicator diagram of E. Thus if $h_n(\theta)$ is the indicator function of $E^{(n)}(z)$, then $h_n(\theta) \leqslant h(\theta)$. In particular, $h_n[\pm(\pi/2)] \leqslant \alpha$ which proves part (a). From the definition of class P,

$$[E(iy)]^{-1} = O(e^{-|y|(\alpha-\epsilon)}), \qquad y \to \pm\infty$$

for a given ϵ, $0 < \epsilon < \alpha$. Thus the integral in (2.1) converges uniformly in the strip $|v| \leqslant \beta$ for $0 < \beta < \alpha - \epsilon$. This proves (b). Now

$$G(u+iv) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{itu}\,\frac{e^{-tv}}{E(it)}\,dt.$$

An integration by parts leads to

$$G(u+iv) = -\frac{1}{2\pi iu}\int_{-\infty}^{+\infty} e^{itu}\,D_t\left[\frac{e^{-tv}}{E(it)}\right]dt.$$

Using part (a) of this theorem and the definition of class P, we obtain the result in part (c) in the case $n = 1$. The general case follows on successive integrations by parts.

THEOREM 2.2. *Let E be an entire function of class P.*

(a) *Equation* (1.2) *holds if $z = iy$, y real,*

(b) *If $(1+x^2)^{-N}\phi(x) \in L(-\infty, \infty)$ for some non-negative integer N, then the function defined in* (1.1) *is analytic in the strip $|v| < \alpha$.*

The fact that (1.2) holds with $z = iy$, y real, is a consequence of a result in Fourier transforms, [5, p. 10f]. The second conclusion follows from the uniform convergence of the integral in (1.1) which is a result of Lemma 2.1, part (c).

Now for $0 \leqslant \rho < 1$, let

$$G(\rho, t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{itx} \frac{E(i\rho x)}{E(ix)} \, dx. \tag{2.2}$$

$G(\rho, t)$ has a number of properties similar to those of $G(t)$ and they are proved in a like manner. In particular we mention that $G(\rho, t)$ exists for every real t and that

$$\frac{E(i\rho x)}{E(ix)} = \int_{-\infty}^{+\infty} e^{-ixt} G(\rho, t) \, dt, \qquad -\infty < x < \infty. \tag{2.3}$$

3. The Main Results

We first establish an important result involving the kernel $G(\rho, t)$.

Theorem 3.1. *Let E be an entire function in class P, and*

$$(1 + x^2)^{-N} \phi(x) \in L(-\infty, \infty)$$

for some non-negative integer N. If f, G, K are defined by Eqs. (1.1), (2.1), *and* (1.6), *respectively, then*

$$\frac{1}{2\pi i} \int_{C_\rho} K(w) f(x + \rho w) \, dw = \int_{-\infty}^{+\infty} G(\rho, x - t) \phi(t) \, dt,$$

where C_ρ is a contour in the w-plane enclosing the indicator diagram of E in its interior and lying in the strip $|v| < \alpha/\rho$, $0 < \rho < 1$.

The existence of C_ρ is assured from the fact that the indicator diagram of E (a compact set) is contained in the strip $|v| < \alpha/\rho$ as a result of the conditions $h[\pm(\pi/2)] = \alpha$.

For w on C_ρ, $x + \rho w$ is in the strip $|v| < \alpha$ and thus in the set where f is analytic. Now

$$\begin{aligned} \frac{1}{2\pi i} \int_{C_\rho} K(w) f(x + \rho w) \, dw &= \frac{1}{2\pi i} \int_{C_\rho} K(w) \, dw \int_{-\infty}^{+\infty} G(x + \rho w - t) \phi(t) \, dt \\ &= \int_{-\infty}^{+\infty} \phi(t) \, dt \cdot \frac{1}{2\pi i} \int_{C_\rho} K(w) \, G(x + \rho w - t) \, dw. \end{aligned} \tag{3.1}$$

This interchange is justified because of the uniform convergence of the inner integral of the first equation on C_ρ by Lemma 2.1. With the use of (2.1), the inner integral in the last equation becomes

$$\frac{1}{2\pi i}\int_{C_\rho} K(w)\,dw \cdot \frac{1}{2\pi}\int_{-\infty}^{+\infty} \exp\,[iu(x+\rho w - t)]\,[E(iu)]^{-1}\,du$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} \exp[iu(x-t)]\,[E(iu)]^{-1}\,du \cdot \frac{1}{2\pi i}\int_{C_\rho} e^{iu\rho w}K(w)\,dw$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} \exp[iu(x-t)]\,\frac{E(i\rho u)}{E(iu)}\,du = G(\rho, x-t).$$

The interchange effected here is justified with the use of the fact that the integral in (2.1) with w replaced by $x+\rho w-t$ converges uniformly on C_ρ by the definition of class P. The substitution of this last result into (3.1) completes the proof of the theorem.

Denote the Fourier transform of $g \in L(-\infty, \infty)$ by $g^\wedge$. Thus

$$g^\wedge(x) = \int_{-\infty}^{+\infty} e^{ixt}g(t)\,dt. \tag{3.2}$$

We now prove

THEOREM 3.2. *Under the conditions and definitions of Theorem 3.1 with $N = 0$ and with the additional assumption that*

$$\phi^\wedge(x) = O(e^{-\eta|x|}), \qquad x \to \pm\infty \tag{3.3}$$

for some $\eta > 0$, then

$$\lim_{\rho\to 1-} \frac{1}{2\pi i}\int_{C_\rho} K(w)f(x+\rho w)\,dw = \phi(x) \tag{3.4}$$

for almost all x.

By Theorem 3.1,

$$I_\rho = \frac{1}{2\pi i}\int_{C_\rho} K(w)f(x+\rho w)\,dw = \int_{-\infty}^{+\infty} G(\rho, x-t)\,\phi(t)\,dt$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty}\phi(t)\,dt\int_{-\infty}^{+\infty} e^{i(x-t)u}\,\frac{E(i\rho u)}{E(iu)}\,du$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{ixu}\,\frac{E(i\rho u)}{E(iu)}\,\phi^\wedge(-u)\,du.$$

The interchange is easily justified. There exists a $\delta > 0$ so that for $1-\delta < \rho < 1$,

$$\frac{E(i\rho u)}{E(iu)} = O(e^{\eta(|u|)/2}), \qquad u \to \pm\infty.$$

This follows from the definition of class P. The hypothesis (3.3) shows that we may use the Lebesgue convergence theorem to get

$$\lim_{\rho \to 1-} I_\rho = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{ixu} \phi\hat{}(-u)\, du = \phi(x)$$

for almost all x by an application of a result in Fourier Analysis, [5, p. 19]. This completes the proof.

By changing the values of ϕ on at most a set of measure zero, we see that the last equality above will hold for all x. It is then not difficult to see from (3.3) that ϕ can be extended into the complex plane to be analytic in the strip $z = x + iy$, $|y| < \eta$. Thus, in effect, we are dealing with analytic functions ϕ in Theorem 3.2.

Although the requirement in (3.3) on $\phi\hat{}$ is severe, the class for which this is so is not negligible. Any function $g(z)$ analytic in a strip $|y| < \delta$ and satisfying for some $\beta > 0$

$$g(z) = O(e^{-\beta|z|}), \qquad |z| \to \infty \tag{3.5}$$

uniformly in any substrip $z = x + iy$, $|y| \leqslant \gamma < \delta$ also satisfies (3.3). This can be seen from

$$g\hat{}(x) = \int_{-\infty}^{+\infty} e^{ix(t+ic)} g(t + ic)\, dt \tag{3.6}$$

for $0 < C < \delta$, which is obtained from (3.2) using Cauchy's theorem and (3.5). From (3.6), we see that $g\hat{}(x) = O(e^{-cx})$ as $x \to \infty$. The corresponding result as $x \to -\infty$ can be obtained by using $-\delta < c < 0$. This class contains as a subclass the class $Z_p{}^p$ $(p > 1)$ of all entire functions g for which there exist positive c_1, c_2 with

$$g(x) = O(e^{-c_1|x|^p}), \qquad g(z) = O(e^{c_2|z|^p}),$$

where $z = x + iy$. This class was used by Gel'fand and Shilov in a well known paper [6] in which they show that $Z_p{}^p$ is dense in $L_2(-\infty, \infty)$. Thus the class of functions for which (3.3) is true is extensive.

More information about $E(z)$ will usually lead to sharper results. To illustrate this fact, we introduce the class P_N, a subclass of P. E is of class P_N for N a non-negative integer if

(i) E is of exponential type with no zero along the imaginary axis,

(ii) $E^{(k)}(iy) = O(e^{\alpha|y|})$, $|y| \to \infty$; $k = 0, 1, \ldots, N$,

(iii) $[E(iy)]^{-1} = O(e^{-\alpha|y|})$, $|y| \to \infty$.

Here α is a positive number depending on the function E. Let

$$\Phi_N(x, t) = [1 + (x - t)^2]^{-N} \phi(t).$$

THEOREM 3.3. *Let E be of class P_{2N}, $(1 + x^2)^{-N} \phi(x) \in L(-\infty, \infty)$ and f, G, K be defined by Eqs.* (1.1), (2.1), *and* (1.6), *respectively. Then if, as a function of u,*

$$\Phi_N^{\wedge}(x, u) \in L(-\infty, \infty),$$

we have

$$\lim_{\rho \to 1-} \frac{1}{2\pi i} \int_{C_\rho} K(w) f(x + \rho w)\, dw = \phi(x),$$

for almost all x.

Here $\Phi_N^{\wedge}(x, u)$ is the the Fourier transform of $\Phi_N(x, t)$. Let

$$T_N(x) = (1 + x^2)^N.$$

Then from Theorem 3.1, we can write

$$I_\rho = \frac{1}{2\pi i} \int_{C_\rho} K(w) f(x + \rho w)\, dw$$

$$= \int_{-\infty}^{+\infty} G_1(\rho, x - t)\, \Phi_N(x, t)\, dt,$$

where

$$G_1(\rho, x - t) = T_N(x - t)\, G(\rho, x - t).$$

The representation of $G(\rho, t)$ from (2.2) leads to

$$G_1(\rho, x - t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} [T_N(iD_u)\, e^{iu(x-t)}] \frac{E(i\rho u)}{E(iu)}\, du,$$

where D_u stands for differentiation with respect to u. Successive integrations by parts lead to

$$G_1(\rho, x - t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{iu(x-t)} \left[T_N(iD_u) \frac{E(i\rho u)}{E(iu)}\right] du.$$

Consequently,

$$I_\rho = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \Phi_N(x, t)\, dt \int_{-\infty}^{+\infty} e^{iu(x-t)} \left[T_N(iD_u) \frac{E(i\rho u)}{E(iu)}\right] du$$

$$= \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{ixu} \Phi_N^{\wedge}(x, -u) \left[T_N(iD_u) \frac{E(i\rho u)}{E(iu)}\right] du.$$

The justification for the interchange follows from the definition of class P_{2N}. We now proceed as in the proof of the previous theorem. Since

$$\Phi_N^{\wedge}(x, u) \in L(-\infty, \infty), \qquad \left| T_N(iD_u) \frac{E(i\rho u)}{E(iu)} \right| \leqslant K, \qquad -\infty < u < \infty,$$

for some constant K, we can use the Lebesgue convergence theorem to get

$$\lim_{\rho\to 1-} I_\rho = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{ixu}\Phi_N^\wedge(x, -u)\,du$$

since

$$\lim_{\rho\to 1-}\left[T_N(iD_u)\frac{E(i\rho u)}{E(iu)}\right] = 1.$$

Another appeal to [5, p. 19] gives us

$$\lim_{\rho\to 1-} I_\rho = \phi(x), \qquad \text{a.e.}$$

The simplest case occurs when $N = 0$ for then $\Phi_0(x, t) \equiv \phi(t)$ and the requirements are that $\phi, \phi^\wedge$ be in $L(-\infty, \infty)$. A set of sufficient conditions in this case is that ϕ be twice differentiable and that

(1) ϕ, ϕ', ϕ'' in $L(-\infty, \infty)$

(2) $\phi(t), \phi'(t)$ be $o(1)$ as $t \to \pm\infty$.

These conditions permit us to integrate by parts twice and deduce that $\phi^\wedge(x) = O(x^{-2})$, $x \to \pm\infty$ and hence that $\phi^\wedge$ is in $L(-\infty, \infty)$.

As an example of the general theory, let

$$E(z) = \cosh(z + 1)\cos z.$$

This function is in P_{2N} for any N and its indicator diagram is the square with vertices at $\pm 1 \pm i$. Thus we may apply Theorem 3.3 in this case.

Of course, the cases treated by Hirschman and Widder in [1, Chap. IX] and [2] will also fit under this more general theory.

4. An Extension

Instead of (iii) in the definition of class P, let us suppose that

$$\lim_{y\to\infty}\frac{1}{y}\log|E(iy)| = \alpha_1, \qquad \lim_{y\to-\infty}\frac{1}{|y|}\log|E(iy)| = \alpha_2. \tag{4.1}$$

Then $\alpha_1 + \alpha_2 \geqslant 0$ by a result in entire functions, [3, p. 76]. We will assume that $\alpha_1 + \alpha_2 = 2\alpha > 0$. Let $\alpha_2 - \alpha_1 = 2\beta$ and

$$E_1(z) = e^{-\beta zi}E(z). \tag{4.2}$$

The indicator diagram of $E_1(z)$ is that of $E(z)$ translated β units parallel to the imaginary axis. Moreover,

$$\lim_{|y|\to\infty}\frac{1}{|y|}\log|E_1(iy)| = \alpha > 0. \tag{4.3}$$

Clearly E_1 is in class P. If f_1, G_1, K_1 bear the same relationship to E_1 as do f, G, K to E as defined by equations (1.1), (2.1), and (1.6), then we have

$$G_1(w) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{itw}[E_1(it)]^{-1}\, dt = G(w + \beta i)$$

and similarly

$$K_1(w) = K(w + i\beta), \qquad f_1(x) = f(x + i\beta).$$

We may then deduce the following result from Theorem 3.2:

$$\lim_{\rho \to 1-} \frac{1}{2\pi i} \int_{C_\rho} K(w + i\beta) f(x + i\beta + \rho w)\, dw = \phi(x)$$

for almost all x in $(-\infty, \infty)$. C_ρ is defined in terms of the indicator diagram of E_1. Thus with a change of variable, we may conclude,

THEOREM 4.1. *Let E be in class P with* (iii) *replaced by* (4.1) *with* $\alpha_1 + \alpha_2 > 0$. *Moreover if*

(1) $\phi \in L(-\infty, \infty$,

(2) $\phi^\wedge(x) = O(e^{-\eta|x|})$, $x \to \pm\infty$ *for some* $\eta > 0$,

then

$$\lim_{\rho \to 1-} \frac{1}{2\pi i} \int_{C'_\rho} K(w) f(x + \rho w)\, dw = \phi(x)$$

for almost all x.

C'_ρ is C_ρ translated β units parallel to the imaginary axis.

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